Wall Solution with Weak Gravity Limit in Five Dimensional Supergravity

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Abstract

In five-dimensional supergravity, an exact solution of BPS wall is found for a gravitational deformation of the massive Eguchi-Hanson non-linear sigma model. The warp factor decreases for both infinities of the extra dimension. Thin wall limit gives the Randall-Sundrum model without fine-tuning of input parameters. We also obtain wall solutions with warp factors which are flat or increasing in one side, by varying a deformation parameter of the potential.

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One of the most interesting models in the brane-world scenario is given by Randall and Sundrum, where the localization of four-dimensional graviton [1] has been obtained by a spacetime metric containing a warp factor $e^{2U(y)}$ which decreases exponentially for both infinities of the extra dimension $y \to \pm\infty$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2U(y)} \eta_{mn} dx^m dx^n + dy^2, \tag{1}$$

where $\mu, \nu = 0, .., 4$, $m, n = 0, 1, 3, 4$ and $y \equiv x^2$. They had to introduce both a bulk cosmological constant and a boundary cosmological constant, which have to be fine-tuned each other. Its supersymmetric (SUSY) version has been worked out [2], and has been argued to help understanding the fine-tuning [3].

Since the Randall-Sundrum model uses the bulk AdS space, it has implications for the AdS/CFT correspondence [4], [5]. For that purpose it is also natural to introduce scalar fields forming a smooth wall (thick brane) in supergravity theories. After an extensive studies of BPS walls in four-dimensional supergravity coupled with chiral scalar multiplets [6], an exact solution of a BPS wall has recently been constructed [7]. Studies of domain wall solutions in gauged supergravity theories in five dimensions revealed that hypermultiplets are needed [8] to obtain warp factors decreasing for both infinities $y \to \pm\infty$ (infra-red (IR) fixed points in AdS/CFT correspondence) which are appropriate for phenomenology. It has been shown that the target space of hypermultiplets in five-dimensional supergravity theory must be quaternionic Kähler manifolds [9] in contrast to the hyper-Kähler target space for five-dimensional global SUSY without gravity. The gravitational deformations have been worked out for massless $\mathcal{N} = 2$ nonlinear sigma models in four dimensions [10], [11]. However, massive models, namely models with potential terms are needed to obtain domain wall solutions. Massive hyper-Kähler nonlinear sigma models without gravity in four dimensions have been constructed in harmonic superspace as well as in $\mathcal{N} = 1$ superfield formulation [12], and have yielded the domain wall solution for the Eguchi-Hanson manifold [13] previously obtained in the on-shell component formulation [14]. Domain walls in massive quaternionic Kähler nonlinear sigma models in supergravity theories have been studied using mostly homogeneous target manifolds. Unfortunately, SUSY vacua in homogeneous target manifolds are not truly IR critical points, but can only be saddle points with some IR directions [15], [16]. Inhomogeneous manifolds and a wall solution have also been constructed [17]. However, these manifolds do not allow a limit of weak gravitational coupling, contrary to the model with an exact solution in four-dimensions [7].

The purpose of our paper is to present an exact BPS domain wall solution in five-dimensional supergravity coupled with hypermultiplets (and vector multiplets). We have obtained a two-parameter family of massive quaternionic nonlinear sigma models which reduces to the Eguchi-Hanson nonlinear sigma model for vanishing gravitational coupling. One of the parameters is the gravitational coupling κ, and the other is an asymmetry parameter a for gravitational deformation of potential terms. Having a smooth limit of vanishing gravitational coupling is very useful to obtain inhomogeneous quaternionic Kähler manifolds and also to use it for brane-world phenomenology. The model has two SUSY vacua as genuine local minima up to a critical value of gravitational coupling

$|a| < 1$ gives a warp factor decreasing for both infinities of extra dimension $y \to \pm\infty$, interpolating two IR fixed points. For $|a| = 1$, the warp factor decreases in one direction, and is flat in the other, interpolating an IR fixed point and flat space. For $|a| > 1$, the warp factor decreases in one direction, and increases in the other, interpolating an IR and a ultra-violet (UV) fixed points. If we take a thin wall limit for $a = 0$, we obtain a bulk cosmological constant and boundary cosmological constant satisfying the necessary relation in Ref.[1] from our scalar field configuration automatically. The relation between two cosmological constants is now realized as a consequence of the solution of dynamical equations rather than a fine-tuning between input parameters, similarly to the BPS wall solution in the four-dimensional supergravity [7]. Thus we have obtained the Randall-Sundrum model as a thin-wall limit of a soliton (domain wall) in five-dimensional supergravity. The four-dimensional graviton should be localized on our wall solution [18].

Our strategy to find a gravitational deformation of nonlinear sigma model is to use the recently obtained off-shell formulation of five-dimensional supergravity (tensor calculus) [19], [20] combined with the quotient method via a vector multiplet without kinetic term and the massive deformation (central charge extension). In the off-shell formulation, we can easily introduce the gravitational coupling to the massive hypermultiplets with linear kinetic term which is interacting with the vector multiplet without kinetic term. By eliminating the vector multiplet after coupling to gravity, we automatically obtain a gravitationally deformed constraint resulting in inhomogeneous quaternionic Kähler nonlinear sigma model with the necessary potential terms. We may call the procedure a *massive* quaternionic Kähler quotient method. If we apply this method to any global $\mathcal{N} = 2$ SUSY model with two (or more) isolated SUSY vacua and wall solutions connecting them, we should obtain a gravitationally deformed inhomogeneous quaternionic manifold and wall solutions at least for small gravitational coupling.

Bosonic action of our model in 5D SUGRA

In global $\mathcal{N} = 2$ SUSY case, a BPS wall solution in four dimensions has been found in the nonlinear sigma model with the Eguchi-Hanson target manifold and a potential originating from a mass term [14], [12]. Inspired by this solution, we consider a nonlinear sigma model of hypermultiplets in five-dimensional supergravity which reduces to the massive Eguchi-Hanson nonlinear sigma model in the limit of vanishing gravitational coupling. For this purpose we use the off-shell formulation of Yang-Mills and hypermultiplet matters coupled to supergravity in five dimensions [19], [20][1]. By using the superconformal tensor calculus [19], one can obtain the off-shell Poincaré supergravity action after fixing the extraneous gauge freedoms of dilatation, conformal supersymmetry and special conformal-boost symmetry [20]. We start with the system of a Weyl multiplet, three hypermultiplets and two $U(1)$ vector multiplets. One of the two vector multiplets has no kinetic term and plays the role of a Lagrange multiplier for hypermultiplets to obtain a curved target manifold. The other vector multiplet serves to give mass terms for hypermultiplets.

After integrating out a part of the auxiliary fields by their on-shell conditions in the

[1]We adopt the conventions of Ref.[19] except the sign of our metric $\eta_{\mu\nu} = diag(-1, +1, +1, +1, +1)$. This induces a change of Dirac matrices and the form of SUSY transformation of fermion.

$$
\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2\kappa^2}R - \frac{1}{4}\left(\partial_\mu W^0_\nu - \partial_\nu W^0_\mu\right)\left(\partial^\mu W^{0\nu} - \partial^\nu W^{0\mu}\right) \\
& -\nabla^a \mathcal{A}^\beta_i d_\beta{}^\alpha \nabla_a \mathcal{A}^i_\alpha - \kappa^2[\mathcal{A}^\beta{}_i d_\beta{}^\alpha \nabla_a \mathcal{A}^j_\alpha]^2 \\
& - \left[-\mathcal{A}_i{}^\gamma d_\gamma{}^\alpha (g_0 M^0 t_0 + M^1 t_1)^2{}_\alpha{}^\beta \mathcal{A}^i_\beta - \frac{\kappa^2}{12}(g_0 M^0)^2 (2\mathcal{A}^{(i}_\alpha d^\alpha{}_\gamma (t_0)^{\gamma\beta} \mathcal{A}^{j)}_\beta)^2\right], \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu \mathcal{A}^\alpha_i &= \partial_\mu \mathcal{A}^\alpha_i - (g_0 W^0_\mu t_0 + W^1_\mu t_1)^\alpha{}_\beta \mathcal{A}^\beta_i, & (3)\\
\mathcal{A}^i{}_\alpha &\equiv \epsilon^{ij}\mathcal{A}_j{}^\beta \rho_{\beta\alpha} = -(\mathcal{A}_i{}^\alpha)^*, & (4)
\end{aligned}
$$

where $d_\alpha{}^\beta = diag(1,1,-1,-1,-1,-1)$, κ is the five-dimensional gravitational coupling, $\mathcal{A}^\alpha_i,\ i=1,2,\ \alpha=1,\ldots,6$ are the scalars in hypermultiplets, and W^0_μ (W^1_μ), M^0 (M^1) and t_0 (t_1) are vector fields, scalar fields and generators of the $U(1)$ vector multiplets with (without) a kinetic term. The gauge coupling of W^0_μ is denoted by g_0. Another gauge coupling g_1 is absorbed into a normalization of W^1_μ in order to drop the kinetic term by taking $g_1 \to \infty$. Hypermultiplet scalars are subject to two kinds of constraints

$$
\begin{aligned}
\mathcal{A}^2 &= \mathcal{A}^\beta_i d_\beta{}^\alpha \mathcal{A}^i_\alpha = -2\kappa^{-2}, & (5)\\
1 g_1^2 \mathcal{Y}^{ij}_1 &\equiv 2\mathcal{A}^{(i}_\alpha d^\alpha{}_\gamma (t_1)^{\gamma\beta}\mathcal{A}^{j)}_\beta = 0. & (6)
\end{aligned}
$$

The constraint (5) comes from the gauge fixing of dilatation, and make target space of hypermultiplets to be a non-compact version of quaternionic projective space, $\frac{Sp(2,1)}{Sp(2)\times Sp(1)}$, combined with the gauge fixing of $SU(2)_R$ symmetry. The constraint (6) is required by the on-shell condition of auxiliary fields of the $U(1)$ vector multiplet without kinetic term, and corresponds to the constraint for Eguchi-Hanson target space in the limit of $\kappa \to 0$.

The third line of (2) is a scalar potential consisting of two terms : the first term arises from the couplings to scalars in vector multiplets and the second term from eliminating the auxiliary fields of the $U(1)$ vector multiplet with kinetic term. The scalar M^0 is fixed as $(M^0)^2 = \frac{3}{2}\kappa^{-2}$ from the requirement of canonical normalizations of the Einstein-Hilbert term and the kinetic term of the gravi-photon W^0_μ for Poincaré supergravity. The scalar M^1 without kinetic term is a Lagrange multiplier, and is found to be

$$
M^1 = -\frac{\mathcal{A}^\gamma_i d_\gamma{}^\alpha (t_0 t_1)_\alpha{}^\beta \mathcal{A}^i{}_\beta}{\mathcal{A}_i{}^\gamma d_\gamma{}^\alpha (t_1)^2_\alpha{}^\beta \mathcal{A}^i{}_\beta} g_0 M^0. \quad (7)
$$

Let us introduce two two-component complex fields ϕ_1 and ϕ_2 to parametrize $\mathcal{A}_i{}^\alpha$ by a matrix with $i=1,2$ as rows and $\alpha=1,\ldots,6$ as columns

$$
\mathcal{A}_i{}^\alpha \equiv \frac{1}{\kappa}\bar{\mathcal{A}}^{-1/2}\begin{pmatrix} 1 & 0 & \kappa\phi_1 & -\kappa\phi_2^* \\ 0 & 1 & \kappa\phi_2 & \kappa\phi_1^* \end{pmatrix} \quad (8)
$$

satisfying the constraint (5) by taking $\bar{\mathcal{A}} = 1 - \kappa^2(|\phi_1|^2 + |\phi_2|^2)$. In this basis, we can choose two $U(1)$ generators as

$$
t_1{}^\alpha{}_\beta = \begin{pmatrix} i\alpha & 0 & 0 & 0 \\ 0 & -i\alpha & 0 & 0 \\ 0 & 0 & i\mathbf{1}_2 & 0 \\ 0 & 0 & 0 & -i\mathbf{1}_2 \end{pmatrix}, \quad t_0{}^\alpha{}_\beta = \begin{pmatrix} ia\alpha & 0 & 0 & 0 \\ 0 & -ia\alpha & 0 & 0 \\ 0 & 0 & -i\sigma_3 & 0 \\ 0 & 0 & 0 & i\sigma_3 \end{pmatrix}, \quad (9)
$$

α in t_1 makes target manifold inhomogeneous generally through the constraint (6), and a special case of $\alpha = 1$ corresponds to a homogeneous manifold of $SU(2,1)/U(2)$ [21]. Here we define $\alpha \equiv \kappa^2\Lambda^3$, where Λ is a real parameter of unit mass dimension. We will show later that this choice of two $U(1)$ generators makes the hypermultiplet part of this model be Eguchi-Hanson sigma model with mass term in the limit of $\kappa \to 0$ for fixed Λ. The kinetic terms of scalars in hypermultiplets are rewritten as

$$\begin{aligned}\frac{1}{2}e^{-1}\mathcal{L}_{kin} = &-\bar{\mathcal{A}}^{-1}[(\partial^\mu\phi_1^*\partial_\mu\phi_1 + \partial^\mu\phi_2^*\partial_\mu\phi_2) - (|\phi_1|^2 + |\phi_2|^2 - \kappa^2\Lambda^6)W_\mu^1 W^{1\mu}] \\ &-\kappa^2\bar{\mathcal{A}}^{-2}[|\phi_2^*\partial_\mu\phi_2 + \phi_1\partial_\mu\phi_1^*|^2 + |\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1|^2],\end{aligned} \tag{10}$$

$$W_\mu^1 = -\frac{\mathcal{A}^i_\gamma d^\gamma{}_\alpha \overleftrightarrow{\partial}_\mu((t_1)^\alpha{}_\beta\mathcal{A}_i{}^\beta)}{2\mathcal{A}^i{}_\gamma d^\gamma{}_\alpha(t_1)^{2\alpha}{}_\beta\mathcal{A}_i{}^\beta} = \frac{i(\phi_1\overleftrightarrow{\partial}_\mu\phi_1^* + \phi_2\overleftrightarrow{\partial}_\mu\phi_2^*)}{2\left(-\kappa^2\Lambda^6 + |\phi_1|^2 + |\phi_2|^2\right)}, \tag{11}$$

where $\phi_1\overleftrightarrow{\partial}_\mu\phi_1^* \equiv \phi_1\partial_\mu\phi_1^* - (\partial_\mu\phi_1)\phi_1^*$. The constraint (6) becomes

$$|\phi_1|^2 - |\phi_2|^2 = \Lambda^3, \qquad \phi_1^*\phi_2 = \phi_2^*\phi_1 = 0. \tag{12}$$

After solving the constraint (see Eq.(14)) and rewriting the kinetic terms (10) by using independent variables, the target metric is found to be a quaternionic extension of the Eguchi-Hanson metric [10], [11]. Since the metric is Einstein, the Weyl tensor is anti-selfdual and the scalar curvature is negative $R = -24\kappa^2$, it is locally a quaternionic manifold [9] for any values of $\kappa \neq 0$.

Potential terms of hypermultiplets become

$$\begin{aligned}\frac{1}{2}e^{-1}\mathcal{L}_{pot} = &-(g_0M^0)^2\bar{\mathcal{A}}^{-1}\left[(-a^2\kappa^2\Lambda^6 + |\phi_1|^2 + |\phi_2|^2) - \frac{[-a\kappa^2\Lambda^6 - (\phi_1\sigma_3\phi_1^* + \phi_2^*\sigma_3\phi_2)]^2}{-\kappa^2\Lambda^6 + |\phi_1|^2 + |\phi_2|^2}\right] \\ &+\frac{\kappa^2}{3}(g_0M^0)^2\bar{\mathcal{A}}^{-2}\left[|\phi_1^*\sigma_3\phi_2 + \phi_2\sigma_3\phi_1^*|^2 + |a\Lambda^3 + (\phi_1\sigma_3\phi_1^* - \phi_2^*\sigma_3\phi_2)|^2\right].\end{aligned} \tag{13}$$

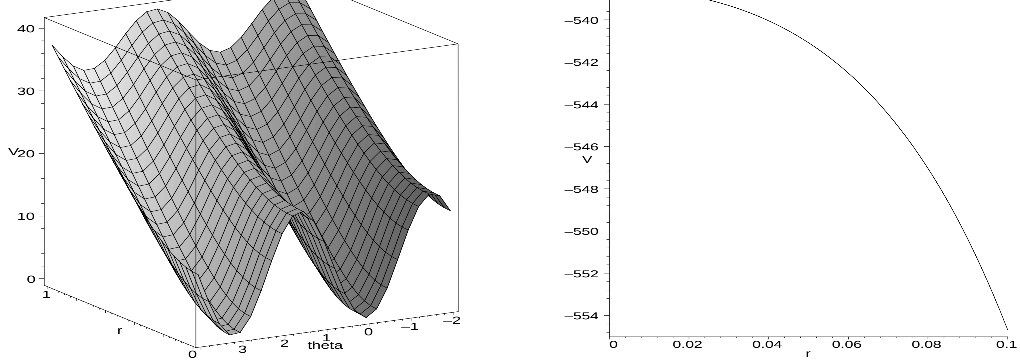
We have now obtained a two-parameter family of gravitational deformations of the Eguchi-Hanson metric by means of the gravitational coupling κ and another deformation parameter a specifying the gravitational deformation of potential terms. This comes about by an asymmetry of $W_\mu^0 t_0$ gauging for the central extension (giving mass terms) relative to the $W_\mu^1 t_1$ gauging for the constraint (producing curved target space of the nonlinear sigma model).

In order to see that the potential (13) has two vacua as local minima, we introduce the spherical coordinates to parametrize two two-component complex fields ϕ_1 and ϕ_2 as

$$\begin{aligned}&\phi_1^1 = g(r)\cos(\tfrac{\theta}{2})\exp(\tfrac{i}{2}(\Psi + \Phi)), \quad &&\phi_1^2 = g(r)\sin(\tfrac{\theta}{2})\exp(\tfrac{i}{2}(\Psi - \Phi)), \\ &\phi_2^1 = f(r)\sin(\tfrac{\theta}{2})\exp(-\tfrac{i}{2}(\Psi - \Phi)), \quad &&\phi_2^2 = -f(r)\cos(\tfrac{\theta}{2})\exp(-\tfrac{i}{2}(\Psi + \Phi)),\end{aligned} \tag{14}$$

where we set

$$f(r)^2 = \frac{1}{2}(-\Lambda^3 + \sqrt{4r^2 + \Lambda^6}), \quad g(r)^2 = \frac{1}{2}(\Lambda^3 + \sqrt{4r^2 + \Lambda^6}), \tag{15}$$



(a) Potential for $a=0$ at $\kappa=0.1$

(b) Potential at $\kappa^2\Lambda^3=0.9$

Figure 1: Discrete vacua. Parameters are taken to be $(g_0M^0,\Lambda)=(3,1)$.

in order to satisfy the constraint (12) [11]. In this coordinate the potential term becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{pot} =& \frac{-2(g_0M^0)^2}{3(1-\kappa^2\sqrt{4r^2+\Lambda^6})^2}\left(v_0+v_1\cos\theta+v_2\cos^2\theta\right), \qquad (16)\\ v_0 =& 3\sqrt{4r^2+\Lambda^6}-\kappa^2(16r^2+3\Lambda^6)-4a^2\kappa^2\Lambda^6\frac{\sqrt{4r^2+\Lambda^6}-3\kappa^2r^2-\kappa^2\Lambda^6}{\sqrt{4r^2+\Lambda^6}-\kappa^2\Lambda^6},\\ v_1 =& -8a\kappa^2\Lambda^3\frac{r^2+\Lambda^6-\kappa^2\Lambda^6\sqrt{4r^2+\Lambda^6}}{\sqrt{4r^2+\Lambda^6}-\kappa^2\Lambda^6},\quad v_2=-\frac{3-2\kappa^2\sqrt{4r^2+\Lambda^6}-\kappa^4\Lambda^6}{\sqrt{4r^2+\Lambda^6}-\kappa^2\Lambda^6}. \end{aligned}$$

The scalar potential $V=-e^{-1}\mathcal{L}_{pot}$ is shown in Fig.1. There exist two vacua at $(r,\theta)=(0,0),(0,\pi)$ as local minima (see Fig. 1-(a)). These two vacua become saddle points with an unstable direction along r for $\kappa^2\Lambda^3>3/4$ for $a=0$. Fig. 1-(b) shows a typical unstable behavior of potential at $\kappa^2\Lambda^3=0.9$, which is close to $\kappa^2\Lambda^3=1$, where the target space of hypermultiplets becomes a homogeneous space of $SU(2,1)/U(2)$. For $a\neq 0$, potential takes different values at these two vacua.

BPS equation

Instead of solving Einstein equations directly, we solve BPS equations to obtain a classical solution conserving a half of SUSY. Since we consider bosonic configurations, we need to examine the on-shell SUSY transformation of gravitino and hyperino [19]

$$\delta_\varepsilon\psi^i_\mu = \mathcal{D}_\mu\varepsilon^i-\frac{\kappa^2}{6}M_0\mathcal{Y}^i_{0j}\gamma_\mu\varepsilon^j, \qquad (17)$$

$$\delta_\varepsilon\zeta^\alpha = -\mathcal{D}_\mu\mathcal{A}^\alpha{}_j\gamma^\mu\varepsilon^j-(M^1t_1+g_0M^0t_0)^\alpha{}_\beta\mathcal{A}^\beta{}_j\epsilon^j+\frac{\kappa^2}{2}\mathcal{A}_j{}^\alpha M_0\mathcal{Y}^j_{0k}\varepsilon^k, \qquad (18)$$

$$\mathcal{D}_\mu \varepsilon^i = (\partial_\mu - \frac{1}{4}\gamma_{ab}\omega_\mu^{ab})\varepsilon^i - \kappa^2 V_\mu{}^i{}_j \varepsilon^j, \tag{19}$$

$$\mathcal{D}_\mu \mathcal{A}_i^\alpha = \partial_\mu \mathcal{A}_i^\alpha + \kappa^2 V_{\mu i}{}^j \mathcal{A}_j^\alpha - W_\mu^1 t_1^\alpha{}_\beta \mathcal{A}_i^\beta, \tag{20}$$

$$\mathcal{Y}_0^{ij} = 2\mathcal{A}_\alpha^{(i} d^\alpha{}_\gamma (g_0 t_0)^{\gamma\beta} \mathcal{A}_\beta^{j)}, \quad V_\mu^{ij} = -\mathcal{A}^{\gamma(i} d_\gamma{}^\alpha \nabla_\mu \mathcal{A}_\alpha^{j)}. \tag{21}$$

If we assume the warped metric (1), the SUSY transformation of the gravitino (17) decouples into two parts

$$\delta_\varepsilon \psi_m^i = \partial_m \varepsilon^i - \frac{1}{2}\gamma_m \gamma^y \partial_y U \cdot \varepsilon^i - \frac{\kappa^2}{6} M^0 \mathcal{Y}_{0j}^i \gamma_m \varepsilon^j, \tag{22}$$

$$\delta_\varepsilon \psi_y^i = \partial_y \varepsilon^i - \kappa^2 V_y^i{}_j \varepsilon^j - \frac{\kappa^2}{6} M^0 \mathcal{Y}_{0j}^i \gamma_y \varepsilon^j. \tag{23}$$

Let us require vanishing of the SUSY variation of gravitino and hyperino to preserve four SUSY specified by

$$\gamma^y \varepsilon^i(y) = i\tau_{3j}^i \varepsilon^j(y), \tag{24}$$

where τ_3 is one of the Pauli matrix. Then one of the gravitino BPS conditions (22) gives an equation for the warp factor $U(y)$ and an additional constraint

$$\partial_y U = \mathcal{W}(\phi) \equiv \frac{2\kappa^2}{3} g_0 M^0 \bar{\mathcal{A}}^{-1}[-a\Lambda^3 - (\phi_1^* \sigma_3 \phi_1 - \phi_2^* \sigma_3 \phi_2)], \tag{25}$$

$$\phi_1^* \sigma_3 \phi_2 = 0. \tag{26}$$

The hyperino BPS condition (18) combined with the condition (26) gives

$$\left[\partial_y - iW_y^1 + \left(\frac{3}{2}\mathcal{W}(\phi) + \bar{V}\right) + (-g_0 M^0 \sigma_3 + M^1)\right](\bar{\mathcal{A}}^{-\frac{1}{2}}\phi_1) = 0,$$

$$\left[\partial_y - iW_y^1 + \left(\frac{3}{2}\mathcal{W}(\phi) - \bar{V}\right) - (-g_0 M^0 \sigma_3 + M^1)\right](\bar{\mathcal{A}}^{-\frac{1}{2}}\phi_2) = 0, \tag{27}$$

$$\bar{V} \equiv \kappa^2 \bar{\mathcal{A}}^{-1}(\phi_1^* \overleftrightarrow{\partial}_2 \phi_1 - \phi_2^* \overleftrightarrow{\partial}_2 \phi_2)/2. \tag{28}$$

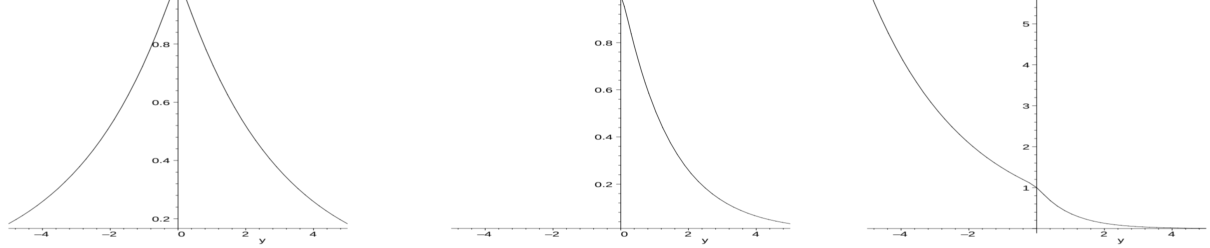
Since Eq.(24) assures that solutions of these BPS equations conserve four SUSY out of eight SUSY, the effective theory on this background has $\mathcal{N} = 1$ SUSY in four dimensions. This should be useful for model building in the SUSY brane-world scenario.

Wall solution and thin wall limit

Let us rewrite the BPS equations in terms of the spherical coordinates (14). After some algebra, we obtain four independent differential equations from Eqs. (27),

$$r\frac{d\Psi}{dy} = 0, \quad \frac{dr}{dy} = \frac{2g_0 M^0 \sqrt{4r^2 + \Lambda^6}}{\sqrt{4r^2 + \Lambda^6} - \kappa^2 \Lambda^6} \cdot r(\cos\theta + a\kappa^2 \Lambda^3),$$

$$\sin\theta \frac{d\Phi}{dy} = 0, \quad \frac{d\theta}{dy} = -2g_0 M^0 \sin\theta. \tag{29}$$



(a) a=0 (b) a=1 (c) a=2

Figure 2: Profile of warped metric. Parameters are taken to be $(g_0 M^0, \Lambda, \kappa) = (3, 2, 0.1)$

Let us obtain the wall solution interpolating between the two vacua: $(r, \theta) = (0, 0), (0, \pi)$. The boundary condition of $r = 0$ at $y = -\infty$ dictates the solution of (29) as

$$r = 0, \quad \cos\theta = \tanh\left(2g_0 M^0 (y - y_0)\right), \quad \Phi = \varphi_0, \tag{30}$$

with Ψ undetermined, and y_0 and φ_0 are constants. Substituting these solutions to r.h.s. of Eq.(25), we obtain the BPS solution of the warp factor

$$U(y) = -\frac{\kappa^2 \Lambda^3}{3(1 - \kappa^2 \Lambda^3)} \left[\ln\{\cosh\left(2g_0 M^0 (y - y_0)\right)\} + 2a g_0 M^0 (y - y_0)\right]. \tag{31}$$

The warp factor $e^{2U(y)}$ of this solution decreases exponentially for both infinities $y \to \pm\infty$ for $|a| < 1$ (see Fig. 2) similarly to the case of the bulk AdS space. Therefore a four-dimensional massless graviton should be localized on the wall [18]. The cases of $|a| = 1$ become the wall solutions interpolating between AdS and flat Minkowski vacua. On the other hand, warp factor increases exponentially either one of the infinities for $|a| > 1$. Following the AdS/CFT conjecture, a vacuum reached by a decreasing (increasing) warp factor corresponds to IR (UV) fixed point of a four-dimensional field theory [4]. Our BPS wall solutions interpolate two IR fixed points for $|a| < 1$. Moreover these vacua are local minima of the potential. This implies that no relevant operator exists in these conformal field theories[2]. The wall solutions for $|a| > 1$ interpolate one IR and one UV fixed points which cannot realize the warped extra dimension, but should be related to a Renormalization Group (RG) flow : the function $\mathcal{W}(\phi)$ in BPS equation of warp factor (25) is monotonic without changing its sign along the flow. The family of our BPS solutions contains a parameter a interpolating between three classes of field theories : one with two IR fixed points ($|a| < 1$), another with one IR and one UV fixed point ($|a| > 1$), and one with one IR fixed point and flat space ($|a| = 1$). We find it remarkable that a single family of models can realize all these possibilities as we change a parameter.

From Eqs.(23) and (24) we find the Killing spinor $\varepsilon^i(y)$ as

$$\varepsilon^i(y) \equiv e^{U(y)/2} \tilde{\varepsilon}^i, \quad \gamma^y \tilde{\varepsilon}^i = i\tau_{3j}^i \tilde{\varepsilon}^j, \tag{32}$$

[2]One of the authors (NS) thanks Steve Gubser for a discussion on this point.

We can obtain a thin wall limit by taking $g_0M^0 \to \infty$ and $\Lambda \to 0$ with $g_0M^0\Lambda^3$, κ, and a fixed. Substituting the solutions (30) and (31) to the Lagrangian of hypermultiplets and taking the thin wall limit, we obtain for $y_0 = 0$

$$
\begin{aligned}
&-\frac{1}{2\kappa^2}R + e^{-1}\mathcal{L}_{kin} + e^{-1}\mathcal{L}_{pot} \\
&= -\frac{1}{2\kappa^2}R + \frac{8\kappa^2(g_0M^0\Lambda^3)^2}{3(1-\kappa^2\Lambda^3)^2}\left[a + \tanh(2g_0M^0y)\right]^2 - 2g_0M^0\Lambda^3(2-\kappa^2\Lambda^3)(1-\kappa^2\Lambda^3)^2 g_0M^0(\cosh(2g_0M^0y)) \\
&\to -\frac{1}{2\kappa^2}R - \left[-\frac{8\kappa^2(g_0M^0\Lambda^3)^2}{3}(a+\epsilon(y))^2\right] - 4(g_0M^0\Lambda^3)\cdot\delta(y), \qquad (33
\end{aligned}
$$

where $\epsilon(y) \equiv \pm 1$ is a sign function. We have obtained a boundary cosmological constant from the wall tension T_w and a bulk cosmological constant $\Lambda_c^+, (\Lambda_c^-)$ for $y < 0 (y > 0)$ as

$$
T_w = 4(g_0M^0\Lambda^3), \qquad \Lambda_c^\pm = -\frac{8\kappa^2(g_0M^0\Lambda^3)^2}{3}(1 \pm a)^2. \qquad (34)
$$

We find that our BPS solution for $a = 0$ automatically satisfies the fine-tuning condition $\sqrt{-\Lambda_c} = \frac{\kappa}{\sqrt{6}}T_w$ of the Randall-Sundrum model between T_w and Λ_c, as a result of combined dynamics of scalar field and gravity. In terms of the asymptotic linear exponent c of the warp factor $U \sim -c|y-y_0|$, $c \equiv 2\kappa^2(g_0M^0\Lambda^3)/3$ for $|y-y_0| \to \infty$, the wall tension $T_w = 24c/(4\kappa^2)$, and cosmological constant $\Lambda_c = -24c^2/(4\kappa^2)$ satisfy precisely the same relation as in Ref.[1] (with $M_p^3 \equiv (4\kappa^2)^{-1}$). Therefore we have realized the single-wall Randall-Sundrum model as a thin-wall limit of our solution of the coupled scalar-gravity theory, instead of an artificial boundary cosmological constant put at an orbifold point.

By a dimensional reduction, we can obtain from the above hypermultiplet action an $\mathcal{N} = 2$ four-dimensional supergravity theory (eight SUSY) with hypermultiplets. Therefore we can automatically obtain from our BPS wall solution (30)-(31) a BPS wall solution in $\mathcal{N} = 2$ four-dimensional supergravity which is a gravitational deformation of the BPS wall solution [14], [12] in the global SUSY case.

Weak gravity limit

Next, we discuss the properties of our model and solution in the weak gravity limit, which is defined by taking the limit of $\kappa \to 0$ with $g_0M^0 \equiv \bar{M}$ held fixed. We obtain in the limit

$$
\begin{aligned}
\frac{1}{2}e^{-1}(\mathcal{L}_{kin} + \mathcal{L}_{pot}) \quad &\to \quad -(\partial^\mu\phi_1^*\partial_\mu\phi_1 + \partial^\mu\phi_2^*\partial_\mu\phi_2) + (|\phi_1|^2 + |\phi_2|^2)W_\mu^1 W^{1\mu} \\
&\quad -\bar{M}^2\frac{(|\phi_1|^2 + |\phi_2|^2))^2 - (\phi_1\sigma_3\phi_1^* + \phi_2^*\sigma_3\phi_2)^2}{|\phi_1|^2 + |\phi_2|^2}, \qquad (35) \\
W_\mu^1 \quad &\to \quad \frac{i(\phi_1\overleftrightarrow{\partial}_\mu\phi_1^* + \phi_2\overleftrightarrow{\partial}_\mu\phi_2^*)}{2(|\phi_1|^2 + |\phi_2|^2)}, \qquad (36)
\end{aligned}
$$

and the constraints (12) are unchanged. The kinetic part in Eq.(35) is identical to the five-dimensional version of the nonlinear sigma model with the target space of $T^*\mathbf{C}P^1$, namely the Eguchi-Hanson manifold, in the basis of Curtright and Freedman [22]. The potential

Ref.[12]: this mass term is originated from the central extension of global $\mathcal{N}=2$ SUSY algebra [12], and can be rewritten as the norm of a tri-holomorphic killing vector for an isometry of target space of the Eguchi-Hanson metric [14]. Therefore the above action (35) has global $\mathcal{N}=2$ SUSY.

In this limit, BPS equations for scalar fields in the hypermultiplets become

$$r\frac{d\Psi}{dy}=0,\quad \frac{dr}{dy}=2\bar{M}r\cos\theta,\quad \sin\theta\frac{d\Phi}{dy}=0,\quad \frac{d\theta}{dy}=-2\bar{M}\sin\theta. \tag{37}$$

These equations are identical to the BPS equations in the massive Eguchi-Hanson sigma model, whose four-dimensional version has been discussed in Ref.[12]. Therefore the model and the solution we discuss in this paper are consistent gravitational deformation of the massive Eguchi-Hanson nonlinear sigma model in five dimensions and associated BPS wall solutions.

The wall solution for $\kappa=0$ is the five-dimensional version of the kink solution in Ref.[14] with the field redefinition in Ref.[12]. Their solutions are exactly identical to our solution (30) obtained for finite κ. It is very interesting that BPS solution for the hypermultiplet ϕ in the global SUSY model coincides with that in the corresponding supergravity. This mysterious coincidence has also appeared in the analytic solution in a four-dimensional $\mathcal{N}=1$ supergravity model [7]. It is tempting to speculate that this property might be related to the exact solvability of our model.

It has been a long-standing problem to find a consistent gravitational deformation from a hyper-Kähler manifold to a quaternionic Kähler manifold with gravitationally corrected potential terms necessary for wall solutions. We have achieved this goal by using an off-shell formulation of supergravity and the *massive* quaternionic Kähler quotient method[3]. Supergravity domain walls have been extensively worked out using the on-shell formulation such as in Ref.[23]. Since auxiliary fields are eliminated when we solve BPS equations, it should in principle be possible to obtain BPS solutions from the on-shell formulation. Off-shell formulation of supergravity, however, offers a more powerful tool to obtain supergravity domain walls as gravitational deformations of those in global SUSY models. If we eliminate constraints before coupling to gravity, it is very difficult in general to extend hyper-Kähler nonlinear sigma models with global eight SUSY to quaternionic Kähler nonlinear sigma models coupled to supergravity, because of the complicated gravitational corrections. On the other hand, many hyper-Kähler sigma models can be obtained as quotients of linear sigma models by using vector multiplets as Lagrange multipliers. When we eliminate Lagrange multiplier multiplets after coupling to gravity in the off-shell formulation, we obtain quaternionic Kähler nonlinear sigma models coupled to supergravity. Moreover we can take a weak gravity limit of these models straightforwardly. Therefore the off-shell formulation of supergravity is quite useful to obtain quaternionic nonlinear sigma models as continuous gravitational deformations of hyper-Kähler nonlinear sigma models of the global SUSY.

As noted in Ref.[11], our quaternionic manifold has a conical singularity at $r=0$ in r,Ψ plane except for discrete values of gravitational coupling $\kappa^2\Lambda^3=(k-1)/k,\ k=2,3,\ldots$

[3]Massless quaternionic Kähler quotient method has been used before [10], [11].

realized for this smooth manifold at least for $\kappa = 2, 3$ ($\kappa = 2$) for $a = 0$ ($|a| > 1$) without having saddle points of the scalar potential. Moreover we believe that we can achieve a continuous gravitational deformation avoiding the singularity, if we simply restore the finite gauge coupling g_1 for the vector multiplet containing W_μ^1 instead of infinite gauge coupling as we did up to now. Let us take a gauged linear sigma model consisting of hypermultiplets interacting with vector multiplets and couple it to supergravity by the tensor calculus [19]. This model is a perfectly consistent interacting supergravity system with eight local SUSY. For finite but large values of gauge coupling g_1, it effectively reduces to our quaternionic nonlinear sigma model except near the conical singularity where we can no longer neglect the vector multiplet. Only in the neighborhood of the singularity, the manifold loses its simple geometrical meaning of quaternionic manifold consisting solely of hypermultiplets. We may call this situation a resolution of the conical singularity[4] in the spirit of Ref.[24]. In this model, we can freely take the limit $\kappa \to 0$ to obtain the Eguchi-Hanson manifold. Therefore we believe that this gauged linear sigma model coupled with supergravity is the most appropriate setting for the gravitational deformation of hyper-Kähler manifolds such as Eguchi-Hanson manifold. On the other hand, our BPS wall should still be a valid solution of the gauged linear sigma model coupled with supergravity. This is because our constraints arising from the elimination of the vector multiplet without kinetic term preserve all SUSY, and hence they solve the BPS condition for the vector multiplet trivially. Therefore we anticipate that our solution continues to be a BPS wall solution for the gauged linear sigma model with a finite large coupling g_1 coupled with supergravity. The only modification should be that the vector multiplet cannot be neglected when we examine the geometry of the target manifold near the resolved conical singularity. We hope to provide a full analysis of the gauged linear sigma model coupled with supergravity in subsequent publications.

Discussion

Finally we discuss implications of our solution on two no-go theorems. It has been shown that wall solutions in supergravity theories always have singularities under several assumptions including non-positive scalar potential [25]. Our BPS wall solution has no singularities of the type they discussed and can be regarded as a counter example of the no-go theorem. This violation of the no-go theorem arises from the fact that a potential becomes positive around the center of the wall contradicting one of their assumptions.

On the other hand, it has been shown that the proposed Nambu-Goldstone (NG) fermion from broken SUSY diverges on the wall in supergravity theories [26]. They considered the result as a no-go theorem for smooth wall solutions such as the one presented here. Recently Cvetic and Lambert have proposed a more proper definition of the wave function of the NG fermion associated with the killing spinor of broken SUSY and have argued that the no-go theorem can be evaded [27]. In fact, we obtained an explicit domain wall solution with warp factor decreasing for both infinities of extra dimensions in five-dimensional supergravity. We regard our result to be an example evading the no-go theorem along the line of Ref.[27].

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